Problem set #10

Problem 1 *Halo potential energy*

Collapsed dark matter halos have a universal density profile first described by Navarro, Frenk, and White (NFW) in 1996. Consider a dark-matter halo with NFW density profile, i.e.

$$\rho(r) = \frac{\rho_{\rm S}}{x\,(1+x)^2} \quad \text{with} \quad x = \frac{r}{r_{\rm S}} \quad . \tag{1}$$

(a) Argue why the total potential energy of the halo must be of the form

$$E_{\rm pot} = -\alpha \frac{G M_{\rm S}^2}{r_{\rm S}} \quad \text{with} \quad M_{\rm S} = 4\pi r_{\rm S}^3 \rho_{\rm S} \quad , \tag{2}$$

where α is a dimension-less constant.

(b) Calculate the enclosed mass $M(<r)$ in terms of x. Confirm that the gravitational potential of a NFW halo is

$$\Phi(r) = -\frac{G M_{\rm S}}{r_{\rm S}} \frac{\ln(1+x)}{x} \quad . \tag{3}$$

(c) Find α by integrating the potential over infinitesimal mass shells, $E_{\rm pot} = \int_0^\infty \Phi(r) {\rm d}M(r)$.

Problem 2 *Halo gas density*

The NFW density profile diverges in the centre, i.e. for $x \to 0$. Gas filled into the halo's gravitational potential Φ satisfies Euler's equation

$$\frac{\vec{\nabla} p_{\rm gas}}{\rho_{\rm gas}} = -\vec{\nabla}\Phi \quad , \tag{4}$$

where $p_{\rm gas}$ is the gas pressure.

(a) Assuming an isothermal and ideal gas, show that the gas density profile is

$$\rho_{\rm gas} = A \exp\left(-\frac{\bar{m}\Phi}{kT}\right) \quad , \tag{5}$$

where T is the temperature, k is Boltzmann's constant, $\bar{m}$ is the mean particle mass, and A is a constant.

(b) Using equation (3) from problem 1, argue that

$$-\frac{\bar{m}\Phi}{kT} = 3\frac{\ln(1+x)}{x} \tag{6}$$

if the gas is in equilibrium with the gravitational potential in the centre. Is the gas density finite in the halo's centre?

Problem 3 *Gravitational Lens Equations*

In the lecture the gravitational lens equation was derived. It maps observed image positions to source positions. The lens equation can be written as

$$\vec{\eta} = \frac{D_\mathrm{s}}{D_\mathrm{d}}\vec{\xi} - D_\mathrm{ds}\vec{\hat{\alpha}}(\vec{\xi}), \tag{7}$$

where $\vec{\eta}$ and $\vec{\xi}$ are vectors in the source and lens/image plane (see Figure 1), D_s, D_d, and D_ds are diameter distances, and $\vec{\hat{\alpha}}(\vec{\xi})$ is the light deflection an observer would measure if (s)he were "sitting in the lens":

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2}\int \Sigma(\vec{\xi}')\frac{\vec{\xi}-\vec{\xi}'}{|\vec{\xi}-\vec{\xi}'|^2}\,\mathrm{d}^2\vec{\xi}'.$$

More relevant is what can be seen on the sky (angular positions) and therefore the lens equation (7) is often transformed into angular coordinates:

$$\vec{\beta} = \vec{\theta} - \frac{D_\mathrm{ds}}{D_\mathrm{s}}\vec{\hat{\alpha}}(\vec{\theta}) \equiv \vec{\theta} - \vec{\alpha}(\vec{\theta}),$$

$\vec{\alpha} = \frac{D_\mathrm{ds}}{D_\mathrm{s}}\vec{\hat{\alpha}}$ is the deflection angle we can observe, $\vec{\beta} = \vec{\eta}/D_\mathrm{s}$ and $\vec{\theta} = \vec{\xi}/D_\mathrm{d}$.
The most simple lens is a point mass, where

$$\vec{\hat{\alpha}}_\mathrm{PM}(\vec{\xi}) = \frac{4GM}{c^2}\frac{\vec{\xi}}{|\vec{\xi}|^2} \equiv 2R_\mathrm{S}\frac{\vec{\xi}}{|\vec{\xi}|^2}$$

holds. The lens equation then reads

$$\vec{\beta} = \vec{\theta} - \frac{D_\mathrm{ds}}{D_\mathrm{s}D_\mathrm{d}}\frac{4GM}{c^2}\frac{\vec{\theta}}{|\vec{\theta}|^2}.$$

Note that this is a two-dimensional vector equation and you can use whatever coordinate system which is convenient to evaluate quantities. Two popular ones are the Cartesian and the polar coordinate systems. Since a point mass is certainly azimuthally symmetric, there exists only one component in the deflection angle (you can think of the radial one, or if you are in Cartesian coordinates, you can choose any as the x-axis of the system).
You obtain a scalar equation

$$\beta = \theta - \frac{D_\mathrm{ds}}{D_\mathrm{s}D_\mathrm{d}}\frac{4GM}{c^2}\frac{1}{\theta}.$$

(a) Solve the quadratic equation for the Einstein angle obtained for $\beta = 0$. Write down the Einstein-*angle* and the Einstein-*radius*, which is the image circle at the lens.

(b) Write down the lens equation in terms of the Einstein angle. When does it have one solution, and when does it have two? Write down the solutions. What does the sign of the solutions mean? Calculate and sketch the separation of the two images.

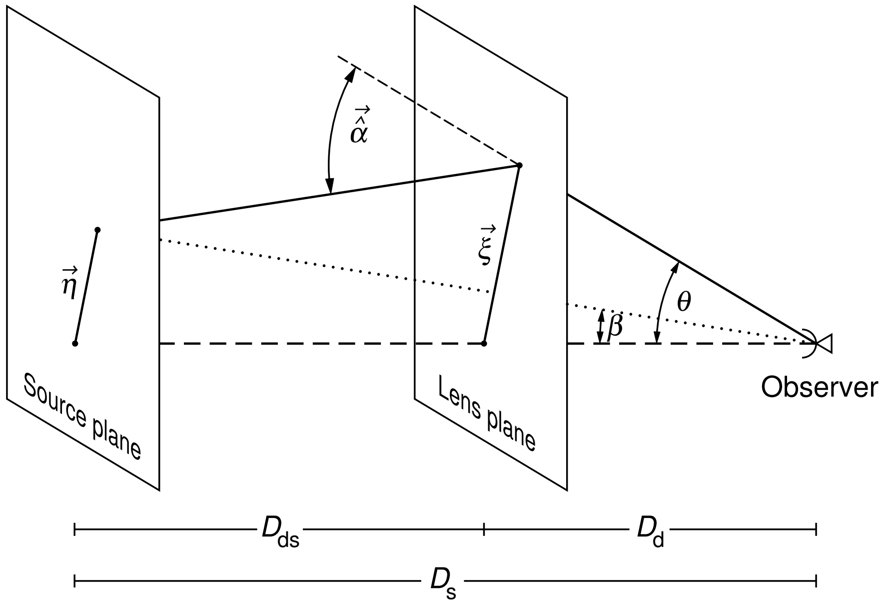

Figure 1: Gravitational lens geometry.