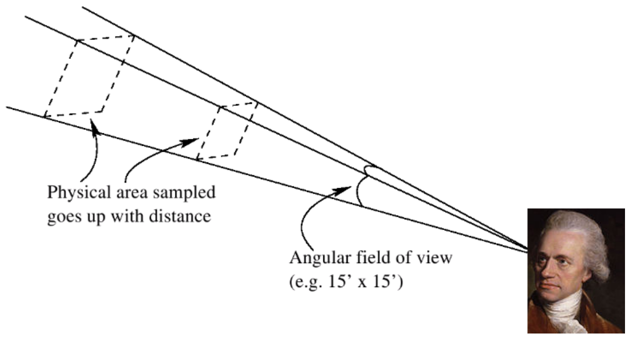
Problem set #4

Problem 1 *William Herschel and the Galaxy*

We measure distances on the sky using angles. Similarly, when we measure the area covered by an image (either a photograph or a digital CCD image), we measure it in units of "square arcminutes" or "square degrees" (one degree is 60 arcminutes, so one square degree is 3600 square arcminutes). The actual volume of space you are sampling in a given picture depends on how far away you are looking.



William Herschel[1] (pictured) was one of the first people to attempt to make a map of the Galaxy (at the time thought to be the whole Universe). He assumed that stars had the same absolute magnitude (now known to be incorrect), and counted how many stars of what magnitudes he could see in various directions to come up with a map of the Galaxy.

Suppose you are William (or Caroline) Herschel looking at a telescope image with a field of view of 15×15 arcminutes^2. Suppose also that you live in the Universe that Herschel thought he lived in: all stars are like the Sun, so that their absolute magnitude is 4.8, and the density of stars is uniform (assume one star per 25 pc^3). Finally, suppose that the images taken have an exposure time that allows to see stars of magnitude 20 and brighter. Ignore extinction.

(a) How far away is the most distant star you see?

(b) How many stars with magnitude between 19 and 20 do you expect to see in your image?

(c) Given this density of stars, what is the brightest star you might expect to see?

(d) Suggest an explanation why William Herschel thought we were at the center of our Galaxy.

(e) Assuming that all the available transmission functions for Herschels' camera are in the right figure of the lecture slide titled "Broadband filters", what filter should Herschel order on Amazon (e.g. Washington T2)?

(f) If the only available one is the Johnson-Cousins U, what do you need to change in the previous calculations (do not solve them again)?

[1] Do not forget his sister Caroline.

Problem 2 *Relations between the equations of state and stellar structure*

With respect to stellar structure the equation of state is in its simplest form represented by the ideal gas law

$$P_{\text{gas}} = nkT.$$

(a) The radiation pressure is given by

$$P_{\text{rad}} = \frac{4\sigma_B T^4}{3c},$$

where σ_B is Stefan-Boltzmann constant. Derive a relation $T(\rho)$ for which gas pressure P_{gas} and radiation pressure P_{rad} become equal. For simplicity, assume the gas is pure hydrogen, therefore $n = \rho/m_p$.

(b) The onset of electron degeneracy can be estimated through the comparison of the Fermi energy to the thermal energy, $E_{\text{gas}} = \frac{3}{2}kT$. Derive a relation $T(\rho)$ for which these energies become equal. Fermi energy is a function of particle number density only.

$$E_f = \frac{\hbar^2}{2m_e}(3\pi^2 n)^{2/3}$$

(c) For higher densities and particularly low temperatures Coulomb interactions of the ions must be considered. The ions tend to form a lattice which minimizes their total energy. The onset of this effect can be estimated through a comparison of thermal energy and Coulomb energy. Derive a relation $T(\rho)$ for which these energies become equal. The mean separation r_{sep} of the ions can be approximated and expressed by the ion density n_{ion}, by making use of $V_{\text{ion}} = (4\pi/3)r_{\text{sep}}^3$ and $V_{\text{ion}} = 1/n_{\text{ion}}$.

(d) Make a plot (pen and paper) of $\log(T)$ (range 10^3 to 10^{10} K) on the y-axis versus $\log(\rho)$ (10^{-7} to 10^7 g cm^{-3}) on the x-axis. Draw the regions where radiation pressure, degeneracy pressure, and Coulomb interactions become strong perturbations. For comparison, also indicate the *central* density and temperature for the Sun.

Problem 3 *The Gamow Energy and the Sun*

The probability that a particle A penetrates the Coulomb barrier of a particle B is given by

$$f_{pen} \simeq \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right] \tag{1}$$

where the Gamow Energy E_G is given by

$$E_G \simeq 2m_r c^2 (\pi\alpha Z_A Z_B)^2. \tag{2}$$

Here α is the fine structure constant and m_r is the reduced mass of the two particles.

(a) Calculate the Gamow energy for the following reactions:

- the collision of two protons;
- the collision of two ${}^3_2$He nuclei. (For simplicity, assume the mass of a ${}^3_2$He nucleus, denoted by m_3, is $3m_\mathrm{p}$).

(b) The core of the Sun has a temperature of 15.6×10^6 K. Calculate the penetration probabilities for the two interactions.

(c) Not all particles within the core of the Sun have the same energy of course. Assume that the fusion probability is actually given by

$$f = \exp\left[-(E_G/E)^{1/2} - \frac{E}{kT}\right] \tag{3}$$

and derive an expression for the location of the Gamow peak. Convert the corresponding energy to temperature.