Problem set #8

Problem 1 *Gravitational potential*

(a) Consider an N-body system of stars with coordinates $\vec{x}_i$ and masses m_i, $i = 1, \ldots, N$. Show that the negative of the gradient of the quantity Φ, defined as

$$\Phi(\vec{x}) = -\sum_{i=1}^{N} \frac{Gm_i}{|\vec{x} - \vec{x}_i|},$$

is equal to the gravitational force of the system. (In other words, Φ is the gravitational potential of the system.)

(b) Newton proved two results that enable us to calculate the gravitational potential of any spherically symmetric distribution of matter easily:

Newton's first theorem: *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*

Newton's second theorem: *The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.*

Consider a spherical distribution of mass with density $\rho(r) \sim r^{-n}$, with $n = \mathrm{d} \ln \rho / \mathrm{d} \ln r$.

- Use Newton's theorems to calculate the gravitational force F on a particle of mass m_1 at a position r_1 for $n = 1, 2, 3, 4$.
- How does the total mass of the system behave when going to large radii ($r_1 \to \infty$)?
- At what value for n does this behaviour change?

Problem 2 *Tully-Fisher relation*

The Tully-Fisher relation is an empirical correlation between the luminosities of disk galaxies (the luminosities of their stellar disks) and their circular velocities. If you assume an exponential surface density profile for the stellar disk, $\Sigma(r) = \Sigma_0 e^{-r/r_\mathrm{d}}$, where r_d gives the typical size of the disk, and a Tully-Fisher relation of the form $L \sim v_\mathrm{max}^4$,

(a) Find the total mass of the disk M as a function of r_d.

(b) How large is the size of a galaxy of mass $\frac{1}{3}M$?

(c) Find the mass-to-light ratio M/L. Use $v_\mathrm{max}^2 = \frac{M}{r_\mathrm{d}}$.

Problem 3 *The black-hole-mass–velocity-dispersion relation*

There is observational evidence that most elliptical galaxies harbor supermassive black holes at their centers. The observations indicate that the black hole masses correlate with global properties of the galaxies. In the table below you will find measurements of the black hole masses $M_{\rm BH}$ and the velocity dispersions σ of the stellar component for a number of elliptical galaxies. Plot the black hole mass against the velocity dispersion and verify that a correlation exists.

TABLE 1
GALAXY SAMPLE

Galaxy	Type	M_B	$M_{\rm BH}$ (Low, High) ($M_\odot$)	Method	σ_1 (km s^{-1})	Distance (Mpc)	M/L, Band	References for Black Hole Masses
Milky Way	SBbc	−17.65	1.8×10^6 (1.5, 2.2)	s, p	103	0.008	1.0, *K*	1
N221 = M32	E2	−15.83	2.5×10^6 (2.0, 3.0)	s, 3I	75	0.81	1.85, *I*	2
N224 = M31	Sb	−19.00	4.5×10^7 (2.0, 8.5)	s	160	0.76	5, *V*	3, 4, 5
N821	E4	−20.41	3.7×10^7 (2.9, 6.1)	s, 3I	209	24.1	5.8, *V*	6, 7
N1023	SB0	−18.40	4.4×10^7 (3.9, 4.9)	s, 3I	205	11.4	5.0, *V*	8
N1068	Sb	−18.82	1.5×10^7 (1.0, 3.0)	m	151	15.0	...	9
N2778	E2	−18.59	1.4×10^7 (0.5, 2.2)	s, 3I	175	22.9	6.4, *V*	6, 7
N2787	SB0	−17.28	4.1×10^7 (3.6, 4.5)	g	140	7.5	...	10
N3115	S0	−20.21	1.0×10^9 (0.4, 2.0)	s	230	9.7	6.9, *V*	11
N3245	S0	−19.65	2.1×10^8 (1.6, 2.6)	g	205	20.9	3.7, *R*	12
N3377	E5	−19.05	1.0×10^8 (0.9, 1.9)	s, 3I	145	11.2	2.7, *V*	6, 13
N3379	E1	−19.94	1.0×10^8 (0.5, 1.6)	s, 3I	206	10.6	4.6, *V*	14
N3384	S0	−18.99	1.6×10^7 (1.4, 1.7)	s, 3I	143	11.6	2.8, *V*	6, 7
N3608	E2	−19.86	1.9×10^8 (1.3, 2.9)	s, 3I	182	22.9	3.7, *V*	6, 7
N4258	Sbc	−17.19	3.9×10^7 (3.8, 4.0)	m, a	130	7.2	...	15
N4261	E2	−21.09	5.2×10^8 (4.1, 6.2)	g	315	31.6	5.0, *V*	16
N4291	E2	−19.63	3.1×10^8 (0.8, 3.9)	s, 3I	242	26.2	4.4, *V*	6, 7
N4342	S0	−17.04	3.0×10^8 (2.0, 4.7)	s, 3I	225	15.3	6.3, *I*	17
N4459	S0	−19.15	7.0×10^7 (5.7, 8.3)	g	186	16.1	...	10
N4473	E5	−19.89	1.1×10^8 (0.31, 1.5)	s, 3I	190	15.7	6.3, *V*	6, 7
N4486 = M87	E0	−21.53	3.0×10^9 (2.0, 4.0)	g	375	16.1	4.0, *V*	18, 19
N4564	E3	−18.92	5.6×10^7 (4.8, 5.9)	s, 3I	162	15.0	1.9, *I*	6, 7
N4596	SB0	−19.48	7.8×10^7 (4.5, 12)	g	152	16.8	...	10
N4649	E1	−21.30	2.0×10^9 (1.4, 2.4)	s, 3I	385	16.8	9.0, *V*	6, 7
N4697	E4	−20.24	1.7×10^8 (1.6, 1.9)	s, 3I	177	11.7	4.8, *V*	6, 7
N4742	E4	−18.94	1.4×10^7 (0.9, 1.8)	s, 3I	90	15.5	...	20
N5845	E3	−18.72	2.4×10^8 (1.0, 2.8)	s, 3I	234	25.9	4.8, *V*	6
N6251	E2	−21.81	5.3×10^8 (3.5, 7.0)	g	290	93.0	8.5, *V*	21
N7052	E4	−21.31	3.3×10^8 (2.0, 5.6)	g	266	58.7	6.3, *I*	22
N7457	S0	−17.69	3.5×10^6 (2.1, 4.6)	s, 3I	67	13.2	3.4, *V*	6, 7
IC 1459	E3	−21.39	2.5×10^9 (2.1, 3.0)	s, 3I	340	29.2	3.1, *I*	23

NOTES.—Distances are taken from Tonry et al. 2001 for most of the galaxies; where these are not available, the distance is determined from the recession velocity, assuming a Hubble constant of 80 km s^{-1} Mpc^{-1}. Absolute magnitudes are for the hot component of the galaxy only. The mass-to-light ratios M/L are usually determined from the same dynamical models that are used to derive the black hole masses; they are given here for reference but play no role in our analysis. Methods: s = stellar radial velocities; p = stellar proper motions; m = maser radial velocities; a = maser accelerations; g = rotating gas disk from emission-line observations; 3I = axisymmetric dynamical models, including three integrals of motion.

REFERENCES.—(1) Chakrabarty & Saha 2001; (2) Verolme et al. 2002; (3) Tremaine 1995; (4) Kormendy & Bender 1999; (5) Bacon et al. 2001; (6) Gebhardt et al. 2002; (7) J. Pinkney et al. 2002, in preparation; (8) Bower et al. 2001; (9) Greenhill & Gwinn 1997; (10) Sarzi et al. 2001; (11) Kormendy et al. 1996a; (12) Barth et al. 2001b; (13) Kormendy et al. 1998; (14) Gebhardt et al. 2000b; (15) Herrnstein et al. 1999; (16) Ferrarese, Ford, & Jaffe 1996; (17) Cretton & van den Bosch 1999; (18) Harms et al. 1994; (19) Macchetto et al. 1997; (20) M. E. Kaiser et al. 2002, in preparation; (21) Ferrarese & Ford 1999; (22) van der Marel & van den Bosch 1998; (23) Cappellari et al. 2002.

Which of the following functions represents the data best?

- $M_{\rm BH} = \alpha\sigma + \beta$
- $M_{\rm BH} = \alpha e^{\beta\sigma}$
- $M_{\rm BH} = \alpha\sigma^{\beta}$
- $M_{\rm BH} = \alpha \log(\beta\sigma)$

Determine the parameters α and β of that function.